# Linear Algebra
## [KOMS119602] - 2022/2023

# 9.2 - General Vector Space

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# Learning objectives

After this lecture, you should be able to:

1. explain the concept of Euclidean vector space;
2. explain the axioms of vector space;
3. analyze if a given set of vectors is a vector space.

 

# **Part 1:** General Vector Space

Now we shall extend the concept of a vector by extracting the most important properties of familiar vectors and turning them into axioms.

Thus, when a set of objects satisfies these axioms, they will automatically have the most important properties of familiar vectors, thereby making it reasonable to regard these objects as new kinds of vectors.

# What is a vector space?

Let $V$ be an arbitrary nonempty set of objects on which two operations are defined:

- addition: a rule associating with each pair of objects in $\mathbf{u}$ and $\mathbf{v}$ and an object in $\mathbf{u} + \mathbf{v}$.
- scalar multiplication: a rule associating with each scalar $k$ and each object $\mathbf{u}$ in $V$, and an object in $k\mathbf{u}$.

$V$ is called a vector space if it satisfies the following six axioms.

# Six axioms of vector space

1. **Closure:** For every $\mathbf{u}, \mathbf{v} \in V$, then:

$$\mathbf{u} + \mathbf{v} \in V \quad \text{and} \quad k\mathbf{u} \in V, \text{for a scalar } k \in \mathbb{R}$$

2. **Commutativity:** For every $\mathbf{u}, \mathbf{v} \in V$, then: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
3. **Associativity:** For every $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, then:

$$\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$$

4. **Identity:** For every $\mathbf{u} \in V$, there exist identity $\mathbf{0}$ and scalar 1, such that:

$$\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u} \quad \text{and} \quad 1\mathbf{u} = \mathbf{u}$$

5. **Inverse:** For every $\mathbf{u} \in V$, there exists $-\mathbf{u} \in V$ such that:

$$\mathbf{u} + (-\mathbf{u}) = (-\mathbf{u}) + \mathbf{u}$$

6. **Distributivity:** For every $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, then:
   - $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$
   - $(k + m)\mathbf{w} = k\mathbf{w} + m\mathbf{w}$
   - $k(m\mathbf{u}) = (km)\mathbf{u}$

# Summarize of axioms

$V$ is a vector space if the followings are satisfied:

For every $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and a scalars $k, m \in \mathbb{R}$, then:

1. $\mathbf{u} + \mathbf{v} \in V$.
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
3. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
4. There exist identity $\mathbf{0}$ and scalar 1, such that:
   $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$ and $1\mathbf{u} = \mathbf{u}$
5. There exists $-\mathbf{u} \in V$ such that: $\mathbf{u} + (-\mathbf{u}) = (-\mathbf{u}) + \mathbf{u}$
6. $k\mathbf{u} \in V$
7. $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$
8. $(k + m)\mathbf{w} = k\mathbf{w} + m\mathbf{w}$
9. $k(m\mathbf{u}) = (km)\mathbf{u}$
10. $1\mathbf{u} = \mathbf{u}$

# So, how to know that $V$ is a vector space?

**Remark.** In the definition of vector space, we do not specify the nature of the vectors and the operations.

For instance, **we do not require that:**

- vectors are in $\mathbb{R}^n$; or
- the operations of addition and scalar multiplications are not necessarily related to the operations $+$ and $\times$ in $\mathbb{R}$.

**Steps to show that a set $V$ is a vector space:**

1. Identify the set $V$ and the objects of $V$ (that will be a vector).
2. Identify the operations "addition" and "scalar multiplication" in $V$.
3. Check that Axiom 1 is *closed* under addition and scalar multiplication.
4. Check the other five axioms.

# **Part 2:** Example

 

# Example of vector space (1)

*$\mathbb{R}^n$ (including $\mathbb{R}^2$ and $\mathbb{R}^3$) with the standard operations $+$ and $\times$, is a vector space.*

- $V = \mathbb{R}^n = \{\mathbf{u} = (u_1, u_2, \ldots, u_n), \text{ where } u_i \in \mathbb{R}\}$
- *Operations:* the addition and scalar multiplication are defined as:

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n)$$
$$k\mathbf{u} = (ku_1, ku_2, \ldots, ku_n)$$

- *Closure*: for every $u, v \in \mathbb{R}^n$ and $k \in \mathbb{R}$, then

$$\mathbf{u} + \mathbf{v} \in \mathbb{R}^n \quad \text{and} \quad k\mathbf{u} \in \mathbb{R}^n$$

- It can be verified that Axiom 2- Axiom 6 are satisfied. Check it!

# Example of vector space (2)

*The space of* $(2 \times 2)$*-matrices in* $\mathbb{R}$ *is a vector space.*

- $V$ is the set of $(2 \times 2)$-matrices with elements in $\mathbb{R}$.
- The addition and scalar multiplication operations are defined as:

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} = \begin{bmatrix} u_{11} + v_{11} & u_{12} + v_{12} \\ u_{21} + v_{21} & u_{22} + v_{22} \end{bmatrix}$$

$$k\mathbf{u} = k \left( \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \right) = \begin{bmatrix} ku_{11} & ku_{12} \\ ku_{21} & ku_{22} \end{bmatrix}$$

- Clearly, $\mathbf{u} + \mathbf{v}$ and $k\mathbf{u}$ are matrices of size $2 \times 2$.
- *Can you verify Axiom 2 - Axiom 6?*

# Example of vector space (2) (*cont.*)

- **Axiom 2 (commutative)**:

$$\underbrace{\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}}_{\mathbf{u}+\mathbf{v}} = \underbrace{\begin{bmatrix} u_{11}+v_{11} & u_{12}+v_{12} \\ u_{21}+v_{21} & u_{22}+v_{22} \end{bmatrix}}_{=} = \underbrace{\begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} + \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}}_{\mathbf{v}+\mathbf{u}}$$

- **Axiom 3 (identity)**: the zero element is $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$\mathbf{0} + \mathbf{u} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \mathbf{u}$$

  also:

$$1\mathbf{u} = 1\left(\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}\right) = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}$$

- **Axiom 4 (inverse)**: For $\mathbf{u} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}$, then $-\mathbf{u} = \begin{bmatrix} -u_{11} & -u_{12} \\ -u_{21} & -u_{22} \end{bmatrix}$.

  So, $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$

- **Axiom 5 (associative)** and **Axiom 5 (distributive)** are also satisfied.

# Example of vector space (3)

**Vector space of real functions**

*Let $V$ be a set of functions in $\mathbb{R}$, namely: $f := \mathbb{R} \to \mathbb{R}$, with addition and scalar multiplication are defined as follows:*

- *For $f = f(x)$ and $g = g(x)$ in $V$ and $k \in \mathbb{R}$:*

$$(f+g)(x) = f(x) + g(x) \text{ and } (kf)(x) = kf(x)$$

*Then $V$ is a vector space.*

- **Closure**: Note that $(f+g)(x)$ and $(kf)(x)$ are functions in $\mathbb{R}$. So, $(f+g), (kf) \in V$.
- **Commutative**: $(f+g)(x) = f(x) + g(x) = g(x) + f(x) = (g+f)(x)$
- **Zero identity** is the function: $z(x) = 0$ (all elements are mapped to 0), and it satisfies:

$$f(x) + 0 = 0 + f(x) = f(x)$$

- The **negative** of $f(x)$ is $-f(x)$, meaning that $-f := x \to -x$, and it satisfies:

$$f(x) + (-f(x)) = 0 = (-f(x)) + f(x)$$

# Example of vector space (4)

**Vector space of polynomials**

*Let $V$ be the set of all polynomials of the form:*

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

*for $x \in \mathbb{R}$, where addition and scalar multiplication are defined as follows:*

*For $\mathbf{p} = p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ and $\mathbf{q} = q(x) = b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n$. Then:*

$$\mathbf{p} + \mathbf{q} = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \cdots + (a_n + b_n)x^n$$

$$k\mathbf{p} = ka_0 + ka_1 x + ka_2 x^2 + \cdots + ka_n x^n$$

*Then, $V$ is a vector space.*

# Example of vector space (4) (*cont.*)

- **Closure**: the addition of two polynomials and scalar multiplication of a polynomial result in a polynomial in $\mathbb{R}$.
- **Commutative**: $p(x) + q(x) = q(x) + p(x)$
- **Identity element**: there is a zero polynomial 0, s.t. $p(x) + 0 = 0 + p(x) = p(x)$.
- the negative of polynomial $p(x)$ is:

$$-p(x) = a_0 - a_1 x - a_2 x^2 - \cdots - a_n x^n$$

so that: $p(x) + (-p(x)) = (-p(x)) + p(x) = 0$.

# Example of non vector space

*Let* $V = \mathbb{R}^2$ *be the set of objects of form:* $\mathbf{u} = (u_1, u_2),\ u_i \in \mathbb{R}$. *Define addition and scalar multiplication as follows:*

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2)$$
$$k\mathbf{u} = (ku_1, k^2 u_2)$$

$V$ is not a vector space.

- Axiom 1 (*closure*) is satisfied, namely:

$$\forall \mathbf{u}, \mathbf{v} \in V,\ k \in \mathbb{R},\quad \mathbf{u} + \mathbf{v} \in V \ \text{ and } \ k\mathbf{u} \in V$$

- But, Axiom 4 (*identity*) is not satisfied, namely:

$$1\mathbf{u} = 1(ku_1, k^2 u_2) \neq \mathbf{u} \ \text{ for some } \mathbf{u} \in V$$

For example, take $\mathbf{u} = (1, 2)$. Then: $1\mathbf{u} = (1, 4) \neq \mathbf{u}$.

 

*to be continued...*